Bombelli's Example

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In approximately the year 1569 Rafael Bombelli considered the cubic equation $x^3 - 15x - 4 = 0$. This equation has a negative discriminant so solving the equation with the cubic formula involves square roots of negative numbers. Bombelli showed that in spite of that the cubic formula yields the correct real answer.

In the book *Complex Analysis for Mathematics and Engineering* authors John Mathews and Russell Howell assert the following:

> [This] was a proverbial bombshell. Prior to Bombelli, mathematicians could easily scoff at imaginary numbers when they arose as solutions to quadratic equations. With cubic equations, they no longer had this luxury. That $x = 4$ was a correct solution to the equation $x^3 - 15x - 4 = 0$ was indisputable, as it could be checked easily. However, to arrive at this very real solution, mathematicians had to take a detour through the uncharted territory of "imaginary numbers." Thus, whatever else might have been said about these numbers (which, today, we call complex numbers), their utility could no longer be ignored.

For the equation $x^3 - 15x - 4 = 0$ the cubic formula requires the evaluation of the following expression.

$$\sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$$

Algebra, including operations involving the square root of negative values and selecting to one's convenience among the choices of a cube root, can be used to reach the result as follows:

$$\begin{aligned}
&\sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}} \\
&= \sqrt[3]{2 + 11\sqrt{-1}} + \sqrt[3]{2 - 11\sqrt{-1}} \\
&= \sqrt[3]{6 + 3\sqrt{-1} + 8\sqrt{-1} - 4} + \sqrt[3]{6 - 3\sqrt{-1} - 8\sqrt{-1} - 4} \\
&= \sqrt[3]{(3 + 4\sqrt{-1})(2 + \sqrt{-1})} + \sqrt[3]{(3 - 4\sqrt{-1})(2 - \sqrt{-1})} \\
&= \sqrt[3]{(4 + 2\sqrt{-1} + 2\sqrt{-1} - 1)(2 + \sqrt{-1})} + \sqrt[3]{(4 - 2\sqrt{-1} - 2\sqrt{-1} - 1)(2 - \sqrt{-1})} \\
&= \sqrt[3]{(2 + \sqrt{-1})(2 + \sqrt{-1})(2 + \sqrt{-1})} + \sqrt[3]{(2 - \sqrt{-1})(2 - \sqrt{-1})(2 - \sqrt{-1})} \\
&= (2 + \sqrt{-1}) + (2 - \sqrt{-1}) \qquad \text{(selecting among the cube roots; not strictly equal to previous line)} \\
&= 4
\end{aligned}$$