The Cubic Formula

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March 26, 2016

The general form of the cubic equation:

$$ax^3 + bx^2 + cx + d = 0 \qquad \text{(a is not zero)} \tag{1}$$

$$\begin{aligned} \text{Let } y &= x + \frac{b}{3a} \\ x &= y - \frac{b}{3a} \end{aligned} \tag{2}$$

Substitute $y - \dfrac{b}{3a}$ for x in (1):

$$a\left(y - \frac{b}{3a}\right)^3 + b\left(y - \frac{b}{3a}\right)^2 + c\left(y - \frac{b}{3a}\right) + d = 0 \tag{3}$$

$$\begin{aligned} y - \frac{b}{3a} &= \frac{3ay - b}{3a} \\ &= \frac{27a^2y - 9ab}{27a^2} \end{aligned} \tag{4}$$

$$\begin{aligned} \left(y - \frac{b}{3a}\right)^2 &= \frac{(3ay - b)(3ay - b)}{(3a)(3a)} \\ &= \frac{9a^2y^2 - 3aby - 3aby + b^2}{9a^2} \\ &= \frac{9a^2y^2 - 6aby + b^2}{9a^2} \\ &= \frac{27a^2y^2 - 18aby + 3b^2}{27a^2} \end{aligned} \tag{5}$$

$$\begin{aligned} \left(y - \frac{b}{3a}\right)^3 &= \frac{\left(9a^2y^2 - 6aby + b^2\right)(3ay - b)}{(9a^2)(3a)} \\ &= \frac{27a^3y^3 - 18a^2by^2 + 3ab^2y - 9a^2by^2 + 6ab^2y - b^3}{27a^3} \\ &= \frac{27a^3y^3 - 27a^2by^2 + 9ab^2y - b^3}{27a^3} \end{aligned} \tag{6}$$

Substitute (4), (5) and (6) in (3):

$$a\left(\frac{27a^3y^3-27a^2by^2+9ab^2y-b^3}{27a^3}\right)+b\left(\frac{27a^2y^2-18aby+3b^2}{27a^2}\right)+c\left(\frac{27a^2y-9ab}{27a^2}\right)+d=0$$

$$\frac{27a^3y^3-27a^2by^2+9ab^2y-b^3}{27a^2}+\frac{27a^2by^2-18ab^2y+3b^3}{27a^2}+\frac{27a^2cy-9abc}{27a^2}+\frac{27a^2d}{27a^2}=0$$

$$\frac{27a^3y^3-9ab^2y+2b^3+27a^2cy-9abc+27a^2d}{27a^2}=0$$

$$\frac{27a^3y^3}{27a^2}+\frac{27a^2cy-9ab^2y}{27a^2}+\frac{2b^3+27a^2d-9abc}{27a^2}=0$$

$$ay^3+\left(\frac{3ac-b^2}{3a}\right)y+\frac{2b^3+27a^2d-9abc}{27a^2}=0$$

$$y^3+\left(\frac{3ac-b^2}{3a^2}\right)y+\frac{2b^3+27a^2d-9abc}{27a^3}=0 \tag{7}$$

$$\text{Let } p=\frac{3ac-b^2}{3a^2} \tag{8}$$

$$\text{Let } q=\frac{2b^3+27a^2d-9abc}{27a^3} \tag{9}$$

Substitute (8) and (9) in (7):

$$y^3+py+q=0 \tag{10}$$

$$\text{Let } u+v=y$$

$$y=u+v \tag{11}$$

Substitute (11) in (10):

$$(u+v)^3+p\,(u+v)+q=0$$

$$(u+v)\,(u+v)\,(u+v)+p\,(u+v)+q=0$$

$$\left(u^2+2uv+v^2\right)(u+v)+p\,(u+v)+q=0$$

$$\left(u^3+2u^2v+uv^2+u^2v+2uv^2+v^3\right)+p\,(u+v)+q=0$$

$$\left(u^3+v^3\right)+\left(2u^2v+uv^2+u^2v+2uv^2\right)+p\,(u+v)+q=0$$

$$\left(u^3+v^3\right)+\left(3u^2v+3uv^2\right)+p\,(u+v)+q=0$$

$$\left(u^3+v^3\right)+3uv\,(u+v)+p\,(u+v)+q=0$$

$$\left(u^3+v^3\right)+(3uv+p)\,(u+v)+q=0 \tag{12}$$

From (12) it is clear that if $3uv = -p$ then

$$u^3 + v^3 = -q \tag{13}$$

If $3uv = -p$ then

$$uv = \frac{-p}{3} \tag{14}$$

Squaring (13) gives:

$$u^6 + 2u^3v^3 + v^6 = q^2 \tag{15}$$

Negative 4 times the third power of (14) gives:

$$-4u^3v^3 = 4\left(\frac{p}{3}\right)^3 \tag{16}$$

Adding (15) and (16) gives:

$$u^6 - 2u^3v^3 + v^6 = 4\left(\frac{p}{3}\right)^3 + q^2$$

$$\left(u^3 - v^3\right)\left(u^3 - v^3\right) = 4\left(\frac{p}{3}\right)^3 + q^2$$

$$u^3 - v^3 = \pm\sqrt{4\left(\frac{p}{3}\right)^3 + q^2} \tag{17}$$

Adding (17) and (13) gives:

$$2u^3 = -q \pm \sqrt{4\left(\frac{p}{3}\right)^3 + q^2}$$

$$u^3 = -\frac{q}{2} \pm \sqrt{\left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2}$$

There are 3 different values for u with the plus choice.

$$u_1 = \sqrt[3]{-\frac{q}{2} + \sqrt{\left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2}} \tag{18}$$

$$u_2 = u_1\left(\frac{-1 + \sqrt{-3}}{2}\right)$$

$$u_3 = u_1\left(\frac{-1 - \sqrt{-3}}{2}\right)$$

Similarly:

$$v^3 = -\frac{q}{2} \mp \sqrt{\left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2}$$

There are 3 different values for v with the minus choice.

$$v_1 = \sqrt[3]{-\frac{q}{2} - \sqrt{\left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2}} \tag{19}$$

$$v_2 = v_1 \left(\frac{-1 + \sqrt{-3}}{2}\right)$$

$$v_3 = v_1 \left(\frac{-1 - \sqrt{-3}}{2}\right)$$

With the combination of these 3 values for u and 3 values for v there would be 9 solutions for $y = u + v$. However, the additional condition $uv = \frac{-p}{3}$ is satisfied only for 3 of the 9 combinations. One of these 3 combinations is $y_1 = u_1 + v_1$. From (18) and (19) values for u_1 and v_1 can be calculated. Substituting these values in (11) gives a value for y which is a root of (10). Substituting the value for y in (2) gives one root of (1):

$$x_1 = u_1 + v_1 - \frac{b}{3a} \tag{20}$$

Let $p' = \frac{p}{3}$

Determine p' from (8):

$$p' = \frac{3ac - b^2}{9a^2} \tag{21}$$

Let $q' = -\frac{q}{2}$

Determine q' from (9):

$$q' = \frac{9abc - 27a^2d - 2b^3}{54a^3} \tag{22}$$

Substitute p' for $\frac{p}{3}$ and q' for $-\frac{q}{2}$ in (18) giving:

$$u_1 = \sqrt[3]{q' + \sqrt{p'^3 + q'^2}} \tag{23}$$

Substitute p' for $\frac{p}{3}$ and q' for $-\frac{q}{2}$ in (19) giving:

$$v_1 = \sqrt[3]{q' - \sqrt{p'^3 + q'^2}} \tag{24}$$

Similarly to (20):

$$x_1 = u_1 + v_1 - \frac{b}{3a} \tag{25}$$

Once root x_1 is calculated the remaining roots can be determined by dividing $(x - x_1)$ into the original cubic equation. The quadratic formula can then be used.

$$x_2 = \frac{-(ax_1 + b) + \sqrt{(ax_1 + b)^2 - 4a\left(ax_1^2 + bx_1 + c\right)}}{2a}$$
$$x_3 = \frac{-(ax_1 + b) - \sqrt{(ax_1 + b)^2 - 4a\left(ax_1^2 + bx_1 + c\right)}}{2a}$$

The discriminant of the general cubic equation is $p'^3 + q'^2$. If the discriminant is ≥ 0 then calculating a root using (21) through (25) is straightforward. If the discriminant is < 0 the cubic formula will still yield a real root but sophisticated algebra will be needed to eliminate the imaginary portions. To avoid this the trigonometric approach below can be used when the discriminant is < 0.

$$\text{Let } \phi = \arccos\left(\frac{q'}{\sqrt{-p'^3}}\right) \qquad \text{see (33) below}$$
$$x_1 = \left(2\sqrt{-p'}\right)\cos\left(\frac{\phi}{3}\right) - \frac{b}{3a} \qquad \text{see (36) below}$$

If $\left(p'^3 + q'^2\right) < 0$ then $p'^3 < 0$ and $p' < 0$. Therefore the square root operations in the two equations above will not result in imaginary values.

For (33) it should be shown that $\frac{q'}{\sqrt{-p'^3}}$ is within the domain of the arccos function. This is shown after (36) below.

One form of the triple angle formula:

$$\cos(3\theta) = 4\cos(\theta)^3 - 3cos(\theta)$$
$$4\cos(\theta)^3 - 3cos(\theta) = \cos(3\theta) \tag{26}$$

$$\text{Let } \left(2\sqrt{\frac{-p}{3}}\right)\cos(\theta) = y$$
$$y = \left(2\sqrt{\frac{-p}{3}}\right)\cos(\theta) \tag{27}$$

Substitute (27) in (10):

$$\left(\left(2\sqrt{\frac{-p}{3}}\right)\cos(\theta)\right)^3 + p\left(\left(2\sqrt{\frac{-p}{3}}\right)\cos(\theta)\right) + q = 0 \tag{28}$$

$$\text{Multiply (28) by } \frac{-4}{\left(2\sqrt{\frac{-p}{3}}\right)^3}:$$

$$-4\cos(\theta)^3 - \frac{4p\cos(\theta)}{\left(2\sqrt{\frac{-p}{3}}\right)^2} - \frac{4q}{\left(2\sqrt{\frac{-p}{3}}\right)^3} = 0$$

$$-4\cos(\theta)^3 - \frac{4p\cos(\theta)}{\frac{-4p}{3}} - \frac{4q}{\left(2\sqrt{\frac{-p}{3}}\right)\left(\frac{-4p}{3}\right)} = 0$$

$$-4\cos(\theta)^3 + (4p\cos(\theta))\left(\frac{3}{4p}\right) + \frac{12q}{8p\sqrt{\frac{-p}{3}}} = 0$$

$$-4\cos(\theta)^3 + 3\cos(\theta) + \frac{3q}{2p\sqrt{\frac{-p}{3}}} = 0 \tag{29}$$

Adding (26) and (29) gives:

$$\frac{3q}{2p\sqrt{\frac{-p}{3}}} = \cos(3\theta)$$

$$3\theta = \arccos\left(\frac{3q}{2p\sqrt{\frac{-p}{3}}}\right)$$

$$\theta = \left(\frac{1}{3}\right)\arccos\left(\frac{3q}{2p\sqrt{\frac{-p}{3}}}\right) \tag{30}$$

Substitute (30) in (27):

$$y = \left(2\sqrt{\frac{-p}{3}}\right)\cos\left(\left(\frac{1}{3}\right)\arccos\left(\frac{3q}{2p\sqrt{\frac{-p}{3}}}\right)\right) \tag{31}$$

Substitute (31) in (2):

$$x = \left(2\sqrt{\frac{-p}{3}}\right)\cos\left(\left(\frac{1}{3}\right)\arccos\left(\frac{3q}{2p\sqrt{\frac{-p}{3}}}\right)\right) - \frac{b}{3a} \tag{32}$$

$$\text{Let } \phi = \arccos\left(\frac{q'}{\sqrt{-p'^3}}\right) \tag{33}$$

Substitute $p' = \dfrac{p}{3}$ and $q' = \dfrac{-q}{2}$ in (33):

$$\phi = \arccos\left(\frac{\frac{-q}{2}}{\sqrt{\left(\frac{-p}{3}\right)^3}}\right)$$

$$\phi = \arccos\left(\frac{\frac{-q}{2}}{\sqrt{\left(\frac{-p}{3}\right)}\left(\frac{-p}{3}\right)}\right)$$

$$\phi = \arccos\left(\frac{3q}{2p\sqrt{\frac{-p}{3}}}\right) \tag{34}$$

Substitute (34) in (32):

$$x = \left(2\sqrt{\frac{-p}{3}}\right)\cos\left(\left(\frac{1}{3}\right)\phi\right) - \frac{b}{3a}$$

$$x = \left(2\sqrt{\frac{-p}{3}}\right)\cos\left(\frac{\phi}{3}\right) - \frac{b}{3a} \tag{35}$$

Substitute $p' = \dfrac{p}{3}$ in (35):

$$x = \left(2\sqrt{-p'}\right)\cos\left(\frac{\phi}{3}\right) - \frac{b}{3a} \tag{36}$$

When $\left(p'^3 + q'^2\right) < 0$ it should be shown that $\frac{q'}{\sqrt{-p'^3}}$ is within the domain of the arccos function so that ϕ can be determined for (33).

$$\begin{aligned}
\text{Assume } \left(p'^3 + q'^2\right) &< 0 \\
q'^2 &< -p'^3 \qquad (37) \\
0 &\leq q'^2 \\
0 &\leq q'^2 < -p'^3 \\
0 &< -p'^3 \\
-p'^3 &> 0 \qquad \left(-p'^3 \text{ is positive}\right)
\end{aligned}$$

Divide (37) by $-p'^3$. (Dividing by a positive value will not change the direction of the inequality sign.)

$$\begin{aligned}
\frac{q'^2}{-p'^3} &< 1 \\
\sqrt{\frac{q'^2}{-p'^3}} &< 1 \\
\frac{|q'|}{\sqrt{-p'^3}} &< 1
\end{aligned}$$

Expand the absolute value.

$$-1 < \frac{q'}{\sqrt{-p'^3}} < 1$$

Therefore $\frac{q'}{\sqrt{-p'^3}}$ is within the domain of the arccos function.